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Border collision bifurcations, snap-back repellers, and chaos

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The normal form for codimension 1 border collision bifurcations of fixed points of discrete time piecewise smooth dynamical systems is considered in the unstable case. We show that in appropriate parameter regions there is a snap-back repeller immediately after the bifurcation, and hence that the bifurcation creates chaos. Although the chaotic solutions are repellers they may explain observations, and this is illustrated through an example.

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Many applications of computer software involve the modeling and control of systems which depend on both discrete and continuous variables, and so a good understanding of the interaction between these components of the system is important. These systems are often called hybrid or embedded systems, and there is a growing literature of both applications and theory [1–4]. From the perspective of dynamical systems a description of the simple bifurcations which can occur in these systems is important, and many examples are now well understood.

The systems considered here are piecewise smooth and have discrete time and continuous variables. There is a switching surface Σ dividing the regions in which the dynamics is determined by smooth maps, and the equations are continuous across Σ. Thus the left and right sides of Σ could be labeled by L and R, respectively, and a discrete variable defined to take values in $\{L,R\}$ according to which side of Σ the continuous variables are at time n. This discrete variable then determines which dynamical system is applied at the next time step.

Perhaps the most obvious codimension 1 bifurcation of such systems occurs if a fixed point (or periodic orbit) of the system is on (or has a point on) the switching boundary Σ. The two-dimensional normal form for this bifurcation was derived in [1,5], and if the switching surface is transformed to be the y axis ($x=0$) then the local evolution with $\mathbf{x}=(x,y)^T$ is

$$\mathbf{x}_{n+1}=\begin{cases}A_L\mathbf{x}+\mathbf{m} & \text{if } x\leqslant 0\\ A_R\mathbf{x}+\mathbf{m} & \text{if } x\geqslant 0\end{cases}, \quad (1)$$

where the matrices A_L and A_R, and the vector $\mathbf{m}$ are defined as

$$A_\alpha=\begin{pmatrix}T_\alpha & 1\\ -D_\alpha & 0\end{pmatrix} \quad \text{and } \mathbf{m}=\begin{pmatrix}\mu\\ 0\end{pmatrix} \quad (2)$$

for $\alpha=L,R$. The constants T_α and D_α are the trace and determinant of the Jacobian of the defining equations evaluated at the bifurcation point on the left and right of Σ, while μ is the bifurcation parameter. If $\mu=0$ then the origin is a fixed point, and this is clearly in Σ. The question for bifurcation theory is what happens close to the origin when $|\mu|$ is small.

If $|D_R|$ and $|D_L|$ are less than 1 then the local dynamics which can occur has been discussed in a number of papers [1,3,6–9]. In this case bifurcations analogous to the standard saddle node are possible, as is a border crossing in which the fixed point simply moves across the boundary. Depending on the values of the other constants, more complicated possibilities occur, with the creation of other periodic orbits and even chaos. Since the determinant of the Jacobian matrix of a map shows how areas are increased or decreased by iteration, the determinant less than 1 cases can be expected to give information about the stable dynamics which can be observed. It might be imagined that the case of a determinant with modulus greater than 1 is either uninteresting or could be obtained from the modulus less than 1 case by reversing time. However, neither of these is the case: if the map is not invertible we cannot simply reverse time, and the dynamics described below is certainly interesting and relevant to some examples.

The existence of complicated (chaotic) dynamics in the case of a determinant with modulus greater than 1 follows from the theory of snap-back repellers [10–13]. Since the systems considered here are not differentiable across the boundary, we give a brief description of how this theory works below. Next we prove the existence of a snap-back repeller and hence unstable chaos in Eq. (1) for appropriate values of the constants defining A_L and A_R and then show how this bifurcation can be observed in an example of a blowout bifurcation.

Let G_L denote the half plane with $x\leqslant 0$ and G_R denote the half plane with $x\geqslant 0$. For a map such as Eq. (1), we will use the notation F_L and F_R to denote the map in G_L and G_R, respectively. Following [11], we say that the map (1) has a *simple snap-back repeller* if, possibly after the transformation $x\to -x$ (which exchanges the roles of L and R), there exists a fixed point $\mathbf{x}_*^R$ in $x>0$ and

(i) the eigenvalues of A_R have modulus strictly greater than one and $D_L\neq 0$;

(ii) there is a point $\mathbf{x}_0^L$ in $x<0$ such that $F_L(\mathbf{x}_0^L)=\mathbf{x}_*^R$;

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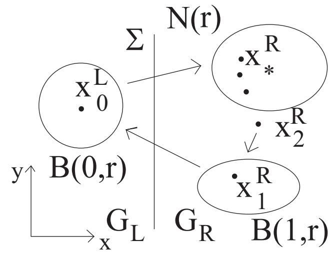

FIG. 1. The geometry of a simple snap-back repeller.

(iii) there exists a sequence $\mathbf{x}_i^R$ in $x>0$ which tends to $\mathbf{x}_*^R$ as $i\to\infty$ such that $F_R(\mathbf{x}_{i+1}^R)=\mathbf{x}_i^R$, $i=1,2,3,\ldots$ and $F_R(\mathbf{x}_1^R)=\mathbf{x}_0^L$.

This is shown in Fig. 1. Note that the first condition implies that $\mathbf{x}_*^R$ is a source with a two-dimensional local unstable manifold and so there can be no conventional homoclinic orbits such as those of the Lozi map.

Of course, more complicated connections are possible, with several passages across the boundary, or more than one path, but this is sufficient for our needs here.

We need to define a set of neighborhoods of the points which define the snap-back repeller and on which the chaotic dynamics can be defined. We start by choosing a closed ball of radius r centered on $\mathbf{x}_1^R$, $B(1,r)$, and take r small enough so that $B(1,r)\subset G_R$, $F_R(B(1,r))=B(0,r)\subset G_L$ and none of the other points $\mathbf{x}_i^R$, $i=2,3,\ldots$, are contained in $B(1,r)$.

Now define $B(2,r)$ to be those points in G_R which map to $B(1,r)$ under one iteration of F_R, and again, possibly reducing the size of r this is a closed set in G_R which does not contain any points in Σ. Define $B(n,r)$ inductively so that $F_R(B(n,r))=B(n-1,r)$ and $B(n,r)$ does not intersect Σ. Note that after a finite number of steps these sets, whose maximum width tends to zero, will be sufficiently close to x_*^R and sufficiently small so that no reduction of r will be necessary.

Let $N(r)=F_L(B(0,r))$, which is a set in G_R (possibly having reduced r to ensure no intersection with Σ) containing x_*^R in its interior. By definition, the sets $B(n,r)$ converge to x_*^R and their maximal diameters tend to zero, so there exists $K>0$ such that $B(k,r)\subset N(r)$ for all $k>K$. By construction $F_L\circ F_R^k(B(k,r))=N(r)$ and $F_L\circ F_R^k$ restricted to $B(k,r)$ is a homeomorphism (in fact, affine). Hence for every $k_1>K$ there exists a closed connected set $B(k,k_1,r)\subset B(k,r)$ such that $F_L\circ F_R^k(B(k,k_1,r))=B(k_1,r)$. The standard induction argument for dynamical systems (using the convergence of nested close sets) implies that for any $M>0$ and any sequence $k_0,k_1,k_2,\ldots$ with $K<k_i<K+M$ there exists a nonempty set $B(k_0,k_1,k_2,\ldots,r)\subset B(k_0,r)$ such that

$$F_L\circ F_R^{k_0}(B(k_0,k_1,k_2,\ldots,r))=B(k_1,k_2,k_3,\ldots,r) \tag{3}$$

and hence that there is an unstable chaotic invariant set containing infinitely many periodic points and uncountably many aperiodic points close to the simple snap-back repeller.

Because the argument is so simple in this restricted case it has been worth rehearsing how the simple snap-back repeller implies chaos, as there has been some controversy about the original idea [10,11], and the system (1) does not formally satisfy all the conditions usually imposed; a formalism which is applicable directly to Eq. (1) can be found in [13].

Now return to the normal form (1). Suppose that T_α and D_α, $\alpha=L,R$, are given. The fixed points of the maps are given by

$$x_*^\alpha=\frac{\mu}{1-T_\alpha+D_\alpha},\quad y_*^\alpha=-D_\alpha x_*^\alpha,\quad \alpha=L,R \tag{4}$$

and $\mathbf{x}_*^R$ exists provided $x_*^R>0$, with a similar inequality for the existence of $\mathbf{x}_*^L$. Given T_α and D_α these inequalities define the sign of μ for which these fixed points exist. The fixed points in G_R and G_L coincide at the origin (on Σ) if $\mu=0$.

The geometry of the map near the boundary controls much of what can be observed. The image of the boundary Σ is the x axis, and since $y_{n+1}=-D_\alpha x_n$, G_R (with $x>0$) is mapped to the upper (respectively lower) half plane if $D_R<0$ (respectively $D_R>0$) and G_L is mapped to the upper (respectively lower) half plane if $D_L>0$ (respectively $D_L<0$). In other words, the images of G_L and G_R overlap if D_R and D_L have opposite signs (one positive and one negative), and do not overlap if they have the same signs. This observation goes a long way towards explaining why boundary crossing occurs if the determinants have the same sign, and more complicated bifurcations can occur otherwise.

To fix ideas we will consider the case

$$D_R>1,\quad D_L<0 \tag{5}$$

and aim to show the existence of a snap-back repeller to x_*^R. For geometric simplicity we will make the further assumption that $\mathbf{x}_R^*$ is an unstable node, so the eigenvalues of A_R are real and distinct and greater than 1. This corresponds to the additional condition

$$T_R>2,\quad T_R^2>4D_R,\quad 1-T_R+D_R>0 \tag{6}$$

which implies that the fixed point in G_R given by Eq. (4) exists if $\mu>0$. Then Eq. (5) implies that the images of G_R and G_L lie in the lower half plane, and hence that there is a preimage of $\mathbf{x}_*^R$ in G_L if $y_*^R<0$, which is automatically satisfied from Eq. (4) as $D_R>0$.

A short calculation using Eq. (1) shows that this point, $\mathbf{x}_0^L=(x_0,y_0)$ in the notation of the previous section, is given by

$$x_0=\frac{D_R}{D_L}x_*^R,\quad y_0=\frac{1}{D_L}(T_RD_L-T_LD_R-D_LD_R)x_*^R. \tag{7}$$

By definition the point $\mathbf{x}_1^R$ of Fig. 1 is a preimage of $\mathbf{x}_0^L$ in $x>0$, and for this preimage to exist we must have $y_0<0$ since the images of both G_R and G_L are in the lower half plane. Since $D_L<0$ and $x_*^R>0$ this implies that the condition

$$T_RD_L-T_LD_R-D_LD_R>0 \tag{8}$$

must hold. In this case $\mathbf{x}_1^R=(x_1,y_1)$ exists and by definition $\mathbf{x}_0^R$ is mapped to it by the map in $x>0$ so

$$x_0=T_Rx_1+y_1+\mu,$$

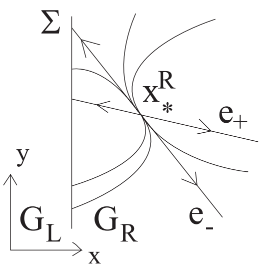

FIG. 2. The geometry of the linear flow in G_R. Solutions lie on curves which (with the exception of $\mathbf{e}_+$) are linear transformations of generalized parabolas $y=x^{\ln s_+/\ln s_-}$.

$$y_0 = -D_R x_1 \tag{9}$$

and hence, using Eqs. (4) and (7),

$$x_1 = -\frac{1}{D_L D_R}(T_R D_L - T_L D_R - D_L D_R)x_*^R,$$

$$y_1 = \frac{1}{D_L D_R}[D_R(D_R - D_L - D_L D_R) + T_R(T_R D_L - T_L D_R)]x_*^R. \tag{10}$$

Note that Eq. (8) ensures that this point does exist in G_R.

Looking back to the definition of a snap-back repeller it remains to show that $\mathbf{x}_1^R$ lies in the two-dimensional unstable manifold of $\mathbf{x}_*^R$, i.e., if $\mathbf{x}_1^R$ is iterated in backwards time using the map in $x>0$ then this orbit remains in $x>0$ and converges to $\mathbf{x}_*^R$.

By Eq. (6) the eigenvalues and eigenvectors of the linear part of the map in $x>0$ are

$$s_\pm = \frac{1}{2}(T_R \pm \sqrt{T_R^2 - 4D_R}), \quad \mathbf{e}_\pm = \begin{pmatrix} s_\pm \\ -D_R \end{pmatrix} \tag{11}$$

with $s_\pm > 1$ and hence the eigenvectors both have negative slopes. Except for solutions on $\mathbf{e}_+$, orbits of the linear map therefore converge in backwards time to $\mathbf{x}_*^R$ on generalized parabolas which are tangential to $\mathbf{e}_-$ (the eigenvalue with smaller modulus) at the fixed point, and this is the eigenvector with the steeper slope. Thus solutions in backwards time lie on curves as sketched in Fig. 2, and clearly all solutions in $y<0$ which start to the left of $\mathbf{e}_+$ tend to the fixed point along solution curves which lie in $x>0$ and $y<0$ for all time and so there will be a simple snap-back repeller.

The remaining (sufficient but not necessary) condition for the snap-back repeller to exist if $T_R>0$ is that $\mathbf{x}_1^R$ lies to the left of $\mathbf{e}_+$. The line of the eigenvector through $\mathbf{x}_*^R$ is $y=-\frac{D_R}{s_+}[x+(s_+-1)x_*^R]$ and so the geometric condition which guarantees the existence of the snap-back repeller is

$$s_+ y_1 \leqslant -D_R[x_1 + (s_+ - 1)x_*^R] \tag{12}$$

which, after some manipulation gives

$$s_+ D_R(D_R - D_L) + (s_+ T_R - D_R)(T_R D_L - T_L D_R) \geqslant 0. \tag{13}$$

In the arguments above we have accumulated a number of conditions: Eqs. (5), (6), (8), and (13) with $T_R>0$. We now need to show that there are some values of the parameters which satisfy all these conditions simultaneously. We start by setting

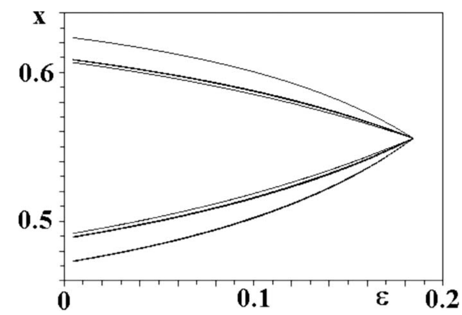

FIG. 3. Bifurcating orbits in the (ϵ, x) plane for the third iterate of the map for the border collision of Eq. (18) with $a=1.8$. Except for the period 3 orbit in $x<a^{-1}$, each orbit has one point in $x>a^{-1}$ and n in $x<a^{-1}$; orbits with $n=0,1,2,3,4$ are shown (periods 3–15 for the map).

$$D_R = 10, \quad T_R = 7 \tag{14}$$

in which case $s_+=5$ and since $1-T_R+D_R=4$ the fixed point exists if $\mu>0$. The final two constraints give

$$-3D_L - 10T_L > 0, \quad 10 + 3D_L - 5T_L > 0. \tag{15}$$

To show how the existence of the snap-back repeller is to some extent independent of the linear type of the flow in G_L we consider briefly several possibilities which satisfy these constraints.

First suppose that

$$T_L = 0, \quad 0 < -D_L < 1. \tag{16}$$

Then both conditions in Eq. (15) are satisfied so the snap-back repeller exists if $\mu>0$. The fixed point in G_L is stable and since $1-T_L+D_L>0$ it exists if $\mu<0$. Hence as μ increases through zero a stable fixed point is destroyed and an unstable fixed point with a strange invariant set from the snap-back repeller is created (possibly with other recurrent dynamics; we have not made an exhaustive study here). If $T_L=-1$ and $D_L=-\frac{1}{4}$ there is a similar bifurcation but in this case the stable fixed point in $\mu<0$ is replaced by a saddle.

Another interesting transformation occurs if

$$D_L = -2, \quad T_L < \frac{3}{5} \tag{17}$$

which satisfies Eq. (15), so the snap-back repeller exists in $\mu>0$. If $-1<T_L<\frac{3}{5}$ then $1-T_L+D_L<0$ and so the fixed point in G_L also exists if $\mu>0$. So in this case we know of no recurrent dynamics if $\mu<0$, but two fixed points and the strange invariant set exist if $\mu>0$. Note that Eq. (17) shows that the snap-back repeller can exist for $\mu>0$ over an unbounded set of the values of the other parameters in the problem.

If $T_R<0$ and both eigenvalues are real and less than minus 1, then the same arguments can be made, and Eq. (6) needs to hold, but the equivalent simple geometric condition to Eq. (13) is less helpful. Of course, given an example it is straightforward to determine whether (x_1,y_1) given by Eq. (10) lies in the two-dimensional unstable manifold of $\mathbf{x}_*^R$ in G_R by backwards iteration of one branch of the map. This allows us to consider the coupled map system introduced in [14] to investigate blowout bifurcations:

$$x_{n+1} = (1-\epsilon)f_a(x_n) + \epsilon f_a(y_n),$$

$$y_{n+1} = \epsilon f_a(x_n) + (1-\epsilon)f_a(y_n), \tag{18}$$

where $\epsilon \in (0,\frac{1}{2})$ and $f_a:[0,1]\to[0,1]$ is the skew tent map

$$f_a(z) = \begin{cases} az & \text{if } z \leq a^{-1} \\ \dfrac{a}{a-1}(1-z) & \text{if } z > a^{-1} \end{cases}, \quad a > 1. \tag{19}$$

The synchronized state $x=y$ is transversely stable provided $\epsilon > \frac{1}{2a}$, but if $\epsilon < \frac{1}{2a}$ nonsynchronized orbits can be created from the two boundaries $x=a^{-1}$ and $y=a^{-1}$. One of the simplest border collisions in this example is for orbits of period 3. If $a=1.8$ and $\epsilon=0.18$, then there are two period 3 orbits, one with a point close to $x=a^{-1}\approx 0.555$ at (0.553, 0.737) and the other with a point at (0.559, 0.736). As ϵ increases these tend to the boundary, one from the left and the other from the right, and there is a border collision at $\epsilon \approx 0.1845$. For the third iterate of the map these are repelling fixed points, and (again for the third iterate) $T_L \approx -9.83$, $D_L \approx 21.77$, $T_R \approx -1.44$, and $D_R \approx -27.21$. Condition (6) is satisfied, and since this is the negative trace case we have checked numerically that the backward orbit of the preimage of the periodic point does indeed tend to the periodic point in backwards time without crossing the border, which shows that the left hand period 3 point (0.553, 0.737) is a snap-back repeller (so the role of left and right are interchanged here). Figure 3 shows some of the bifurcating orbits. Following the blowout bifurcation these orbits become part of the attractor for the system, so their creation and existence are important to understanding the eventual attractor of Eq. (18).

In this Rapid Communication we have shown that snap-back repellers exist in the normal form for unstable border collision bifurcations, which makes it possible to predict the existence of chaotic solutions. These solutions are repelling, but can help explain the existence of periodic and aperiodic orbits which play an important role in the dynamics of the system as described in [15]. A fuller description of the cases will be given elsewhere.

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